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# On the number of combinations without certain separations

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**Abstract**

In this paper we enumerate the number of ways of selecting $k$ objects from $n$ objects arrayed in a line such that no two selected ones are separated by $m-1, 2m-1, \ldots, pm-1$ objects and provide three different formulas when $m, p \geq 1$ and $n \geq pm(k-1)$. Also, we prove that the number of ways of selecting $k$ objects from $n$ objects arrayed in a circle such that no two selected ones are separated by $m-1, 2m-1, \ldots, pm-1$ objects is given by $\frac{n}{n-pk}\binom{n-pk}{k}$, where $m, p \geq 1$ and $n \geq mpk+1$.


## 1. Introduction

In 1943, Kaplansky [6] published a recursive derivation of the number of combinations of $n$ objects taken $k$ at a time without two selected ones being consecutive (see also Comtet [2], Riordan [8] and Ryser [9]). In 1981, Konvalina [7] derived the number of combinations of $n$ objects taken $k$ at a time without two selected ones having unit separation, i.e., having exactly one object between them.

Let $[n]$ (resp. $[\overline{n}]$) be the set of $n$ objects $x_1, x_2, \ldots, x_n$ arrayed in a line (resp. circle). Given a subset $N$ of the set $\mathbb{N}$ of nonnegative integers, a subset $A$ of $[n]$ or $[\overline{n}]$ will be called *$N$-separate* if any two objects in $A$ have exactly $j$ objects between them, then $j \in N$. Let $N_m^p = \mathbb{N} - \{m-1, 2m-1, \ldots, pm-1\}$ for any integers $m, p \geq 1$, define $\mathcal{H}_{p,n}^{m,k}$ (resp. $\mathcal{G}_{p,n}^{m,k}$) to be the number of $N_m^p$-separate $k$-subsets of $[n]$ (resp. $[\overline{n}]$). Thus, by our notation, Konvalina [7]

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considered the special case of $N_2^1$-separation of $[n]$ and $[\overline{n}]$, Kaplansky [6] discussed the special case of $N_1^p$-separation of $[n]$ and $[\overline{n}]$, and obtained that

$$\mathcal{H}_{p,n}^{1,k} = \binom{n-p(k-1)}{k} \quad \text{and} \quad \mathcal{G}_{p,n}^{1,k} = \frac{n}{n-pk}\binom{n-pk}{k}. \tag{1.1}$$

In this paper, by combinatorial analysis together with the algebraic method, we extend the above results to the general case of $m$.

## 2. Some preliminary remarks

Let $n = rm + \ell$ with $1 \le \ell \le m$, and let $A_1, \ldots, A_m$ be a partition of $[n] = \{x_1, x_2, \ldots, x_n\}$ into $m$-subsets defined by

$$\begin{aligned} &A_i = \{x_i, x_{m+i}, \ldots, x_{rm+i}\}, \quad 1 \le i \le \ell, \\ &A_i = \{x_i, x_{m+i}, \ldots, x_{(r-1)m+i}\}, \quad \ell+1 \le i \le m, \end{aligned}$$

then putting them in an array,

$$\begin{array}{ccccc} x_1 & x_{m+1} & \cdots & x_{(r-1)m+1} & x_{rm+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_l & x_{m+l} & \cdots & x_{(r-1)m+l} & x_{rm+l} \\ x_{l+1} & x_{m+l+1} & \cdots & x_{(r-1)m+l+1} & \\ \vdots & \vdots & \vdots & \vdots & \\ x_m & x_{2m} & \cdots & x_{(r-1)m+m} & \end{array}$$

For any $k$-subset $B$ of $[n]$, define $B_i = B \cap A_i$. Note that, in the line case, $B$ is $N_m^p$-separate if and only if each $B_i$ is $N_1^p$-separate. From this critical observation together with (1.1), we can obtain the following result.

**Proposition 2.1.** *For any integer $p, m \ge 1$ and $n, k \ge 0$,*

$$\mathcal{H}_{p,n}^{m,k} = \sum_{\sigma_1(k,m)} \prod_{i=1}^{m} \binom{|A_i| - p(k_i - 1)}{k_i}, \tag{2.1}$$

*where $|A_i|$ is the cardinality of the set $A_i$, and $\sigma_1(k,m)$ denotes the all nonnegative integer solutions of $k_1 + k_2 + \cdots + k_m = k$ such that $k_i \le 1 + \frac{|A_i|}{p}$ for $i = 1, 2, \ldots, m$.*

In the next section, we can find the explicit formula for $\mathcal{H}_{p,n}^{m,k}$, and show that when $n$ is large enough ($n \ge mp(k-1)$ here), then $\mathcal{H}_{p,n}^{m,k}$ is independent of the composition of $n$, i.e., $|A_1| + |A_2| + \cdots + |A_m| = n$. However, in the circle case, the above decomposition does not work, for example, when $n = 5, p = 1, m = k = 2$, then $[\overline{5}] = \{x_1, x_2, \ldots, x_5\}$ has five $N_2^1$-separate 2-subsets, which are $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_1\}$, while $\{x_5, x_1\} \cap \{x_1, x_3, x_5\} = \{x_5, x_1\}$ is not an $N_1^1$-separate 2-subsets of $\{x_1, x_3, x_5\}$. In spite of this, we can derive a recurrence relation between $\mathcal{H}_{p,n}^{m,k}$ and $\mathcal{G}_{p,n}^{m,k}$ for $n \ge mpk + 1$.

Given any $N_m^p$-separate $k$-subset $B$ of $[\overline{n}]$, for some $0 \le j \le m$, there exist $j$ elements of $B$, say $x_{i_1}, x_{i_2}, \ldots, x_{i_j}$, lying in the subset $\{x_1, x_2, \ldots, x_{mp}\}$, in other words, each of which is respectively one of the first $p$ objects of $A_{\ell_1}, A_{\ell_2}, \ldots, A_{\ell_j}$, then there are $\binom{m}{j} p^j$ ways to do this. Now delete the related $j(2p+1)$ objects of $[\overline{n}]$, and delete the remainder $p(m-j)$ elements of $\{x_1, x_2, \ldots, x_{mp}\}$, then we get $m$ object sets $A'_1, A'_2, \ldots, A'_m$ in which all elements are arrayed in a line and there are totally $n - p(m-j) - j(2p+1) = n - pm - pj - j$ elements. Note that the condition $n \geq mpk + 1$ leads to $n - pm - pj - j \geq mp(k - j - 1)$, which makes the restricted inequality condition of Eq. (2.1) in Proposition 2.1 redundant. Then there are $\mathcal{H}^{m,k-j}_{p,n-pm-pj-j}$ ways to select the other $k - j$ objects from $A'_1, A'_2, \ldots, A'_m$. Hence, we have

**Proposition 2.2.** *For any integers $p, m \geq 1$, $n, k \geq 0$ and $n \geq mpk + 1$,*

$$\mathcal{G}^{m,k}_{p,n} = \sum_{j \geq 0} \binom{m}{j} p^j \mathcal{H}^{m,k-j}_{p,n-pm-pj-j}. \tag{2.2}$$

Clearly, we can easily compute special values for $\mathcal{H}^{m,k}_{p,n}$ and $\mathcal{G}^{m,k}_{p,n}$, that is,

- $\mathcal{H}^{m,k}_{p,n} = \mathcal{G}^{m,k}_{p,n} = 0$ for $n < k$;
- $\mathcal{H}^{m,0}_{p,n} = \mathcal{G}^{m,0}_{p,n} = 1$;
- $\mathcal{H}^{m,1}_{p,n} = \mathcal{G}^{m,1}_{p,n} = n$ for $n \geq 1$;
- $\mathcal{H}^{m,k}_{p,n+k} = 0$ for $im + 1 \leq k \leq (i+1)m$, $0 \leq n < imp$ and $i \geq 1$;
- $\mathcal{G}^{m,k}_{p,n+k} = 0$ for $im + 1 \leq k \leq (i+1)m$, $0 \leq n < (i+1)mp$ and $i \geq 1$.

Define $\mathcal{H}^{m,k}_{p,n} = \mathcal{G}^{m,k}_{p,n} = 0$ for $k < 0$ or $n < 0$.

## 3. Main result

In order to give explicit formulas for $\mathcal{H}^{m,k}_{p,n}$ and $\mathcal{G}^{m,k}_{p,n}$, we need the following critical lemma.

**Lemma 3.1.** *Let $\lambda_1, \lambda_2, \ldots, \lambda_m, \mu$ be any $m+1$ complex numbers and $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_m$. Define*

$$\Omega^{m,k}_{\mu,\lambda}(\lambda_1, \lambda_2, \ldots, \lambda_m) = \sum_{\sigma(k,m)} \prod_{i=1}^{m} \binom{\lambda_i + \mu k_i}{k_i},$$

$$\Phi^{m,k}_{\mu,\lambda}(\lambda_1, \lambda_2, \ldots, \lambda_m) = \sum_{\sigma(k,m)} \prod_{i=1}^{m} \frac{\lambda_i}{\lambda_i + \mu k_i} \binom{\lambda_i + \mu k_i}{k_i},$$

*where $\sigma(k, m)$ denotes the all nonnegative integer solutions of $k_1 + k_2 + \cdots + k_m = k$.*

*Then for all $m \geq 1$ and $n, k \geq 0$,*

$$\Omega^{m,k}_{\mu,\lambda}(\lambda_1, \lambda_2, \ldots, \lambda_m) = \sum_{j \geq 0} \binom{m+j-2}{j} \binom{\lambda + \mu k + m - 1}{k - j} (\mu - 1)^j, \tag{3.1}$$

$$= \sum_{j \geq 0} \binom{\lambda + (\mu - 1)k + j}{j} \binom{\lambda + \mu k + m - 1}{k - j} (1 - \mu)^j \mu^{k-j}, \tag{3.2}$$

$$= \sum_{j \geq 0} \frac{\lambda + \mu(m+j)}{k} \binom{m+j-1}{j} \binom{\lambda + \mu k + m - 1}{k - j} (\mu - 1)^j, \tag{3.3}$$

$$\Phi_{\mu,\lambda}^{m,k}(\lambda_1, \lambda_2, \ldots, \lambda_m) = \frac{\lambda}{\lambda+\mu k}\binom{\lambda+\mu k}{k}. \tag{3.4}$$

**Proof.** First we recall the definition of the residue of a function. Let $z_0$ be any isolated singular point of a function $f$. Then there is a Laurent series $f(z) = \sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$ valid for $0 < |z-z_0| < R$, for some positive $R$. The coefficient $a_{-1}$ of $(z-z_0)^{-1}$ is called the residue of $f$ at $z_0$, and is usually written $\mathrm{Res}_{z=z_0} f$ (for computing and properties of the residue see for example [3,4]). For simplicity, we write $\mathrm{Res}_z f$ instead $\mathrm{Res}_{z=0} f$.

Note that the generalized binomial coefficient $\binom{\lambda}{k}$ has an integral representation,

$$\binom{\lambda}{k} = \operatorname*{Res}_x \frac{(1+x)^\lambda}{x^{k+1}},$$

which yields that

$$\frac{\lambda}{\lambda+\mu k}\binom{\lambda+\mu k}{k} = \operatorname*{Res}_x \frac{(1+x)^{\lambda+\mu k-1}(1-(\mu-1)x)}{x^{k+1}}. \tag{3.5}$$

Then we have

$$\begin{aligned}
\Omega_{\mu,\lambda}^{m,k}(\lambda_1, \lambda_2, \ldots, \lambda_m) &= \operatorname*{Res}_x \frac{1}{x^{k+1}} \prod_{i=1}^{m} \sum_{k_i \geq 0} x^{k_i} \operatorname*{Res}_{y_i} \frac{(1+y_i)^{\lambda_i+\mu k_i}}{y_i^{k_i+1}}, \\
&= \operatorname*{Res}_x \left\{ \prod_{i=1}^{m} \left. \frac{(1+y_i)^{\lambda_i+1}}{1-(\mu-1)y_i} \right|_{y_i = x(1+y_i)^\mu} \right\} x^{-k-1}, \\
&= \operatorname*{Res}_x \frac{(1+\varphi(x))^{\lambda+m}}{(1-(\mu-1)\varphi(x))^m} x^{-k-1},
\end{aligned}$$

where $\varphi(x) = x(1+\varphi(x))^\mu$. Using the Lagrange inversion formula for $k \geq 1$ and replacing $x$ by $\frac{y}{(1+y)^\mu}$, we get that

$$\begin{aligned}
\Omega_{\mu,\lambda}^{m,k}(\lambda_1, \lambda_2, \ldots, \lambda_m) &= \sum_{j\geq 0} \frac{\lambda+\mu(m+j)}{k} \binom{m+j-1}{j}\binom{\lambda+\mu k+m-1}{k-j}(\mu-1)^j, \\
&= \operatorname*{Res}_y \frac{(1+y)^{\lambda+\mu k+m-1}}{(1-(\mu-1)y)^{m-1}} y^{-k-1}, \\
&= \sum_{j\geq 0} \binom{m+j-2}{j}\binom{\lambda+\mu k+m-1}{k-j}(\mu-1)^j, \\
&= \operatorname*{Res}_y (1-(\mu-1)y)^{\lambda+\mu k}\left(1+\frac{\mu y}{1-(\mu-1)y}\right)^{\lambda+\mu k+m-1} y^{-k-1}, \\
&= \sum_{j\geq 0} \binom{\lambda+(\mu-1)k+j}{j}\binom{\lambda+\mu k+m-1}{k-j}(1-\mu)^j \mu^{k-j}.
\end{aligned}$$

Similarly, we also have

$$\Phi_{\mu,\lambda}^{m,k}(\lambda_1, \lambda_2, \ldots, \lambda_m) = \operatorname*{Res}_x \frac{1}{x^{k+1}} \prod_{i=1}^{m} \sum_{k_i \geq 0} x^{k_i} \operatorname*{Res}_{y_i} \frac{(1+y_i)^{\lambda_i+\mu k_i-1}(1-(\mu-1)y_i)}{y_i^{k_i+1}},$$

$$= \operatorname*{Res}_x \frac{(1+x)^{\lambda+\mu k-1}(1-(\mu-1)x)}{x^{k+1}},$$
$$= \frac{\lambda}{\lambda+\mu k}\binom{\lambda+\mu k}{k}.$$

This completes the proof. $\square$

**Remark 3.2.** Note that Hwang and Wei [5] considered the special case

$$\Omega^{m,k}_{-1,n+m}(n_1+1,\dots,n_m+1) = \sum_{\sigma(k,m)}\prod_{i=1}^{m}\binom{n_i+1-k_i}{k_i},$$

with $n = n_1+n_2+\cdots+n_m$ and obtained its other expression by recurrence relation,

$$\Omega^{m,k}_{-1,n+m}(n_1+1,\dots,n_m+1) = \sum_{j\ge 0}\binom{m+j-2}{j}\binom{n+1-k-2j}{k-2j},$$

which can be derived easily from the proof of Lemma 3.1 if one notices that

$$\Omega^{m,k}_{-1,n+m}(n_1+1,\dots,n_m+1) = \operatorname*{Res}_y \frac{(1+y)^{n+2m-k-1}}{(1+2y)^{m-1}}y^{-k-1}$$
$$= \operatorname*{Res}_y \frac{(1+y)^{n-k+1}}{(1-\frac{y^2}{(1+y)^2})^{m-1}}y^{-k-1}.$$

Also, the Eq. (3.4) is a generalization of Gould's identity [1,2], that is,

$$\sum_{k=0}^{n}\frac{a}{a+ck}\binom{a+ck}{k}\frac{b}{b+c(n-k)}\binom{b+c(n-k)}{n-k} = \frac{a+b}{a+b+ck}\binom{a+b+ck}{k}.$$

Then (3.4) can be proved again by repeatedly using Gould's identity.

Notice that when $n \ge pm(k-1)$ in (2.1), then the inequality condition for $\sigma_1(k,m)$ (i.e., $k_i \le 1+\frac{|A_i|}{p}$) is redundant. Hence, setting $\lambda_i = |A_i|+p$, $\mu = -p$ in (3.1)–(3.3), and combining with Proposition 2.1, we obtain our main result.

**Theorem 3.3.** *Let $p, m, k \ge 1$ be any integers. For $n \ge pm(k-1)$,*

$$\mathcal{H}^{m,k}_{p,n} = \sum_{j\ge 0}\binom{m+j-2}{j}\binom{n+mp+m-pk-1}{k-j}(-p-1)^j,$$
$$= \sum_{j\ge 0}\binom{n+mp-(p+1)k+j}{j}\binom{n+mp+m-pk-1}{k-j}(p+1)^j(-p)^{k-j},$$
$$= \sum_{j\ge 0}\frac{n-pj}{k}\binom{m+j-1}{j}\binom{n+mp+m-pk-1}{k-j}(-p-1)^j,$$

*and for $n \ge mpk+1$,*

$$\mathcal{G}^{m,k}_{p,n} = \frac{n}{n-pk}\binom{n-pk}{k}. \tag{3.6}$$

**Proof.** It just needs to prove (3.6). For $n \geq mpk + 1$, by (2.2), we have

$$\begin{aligned}
\mathcal{G}_{p,n}^{m,k} &= \sum_{j\geq 0} \binom{m}{j} p^j \mathcal{H}_{p,n-pm-pj-j}^{m,k-j} \\
&= \sum_{j\geq 0} \binom{m}{j} p^j \operatorname*{Res}_y \frac{(1+y)^{n-p(k-j)+m-1-pj-j}}{(1+(p+1)y)^{m-1}} y^{-(k-j)-1} \\
&= \operatorname*{Res}_y \frac{(1+y)^{n-pk+m-1}}{(1+(p+1)y)^{m-1}} y^{-k-1} \sum_{j\geq 0} \binom{m}{j} p^j \left\{\frac{y}{1+y}\right\}^j \\
&= \operatorname*{Res}_y \frac{(1+y)^{n-pk-1}(1+(p+1)y)}{y^{k+1}} \\
&= \frac{n}{n-pk} \binom{n-pk}{k},
\end{aligned}$$

which follows by (3.5).

The formulas (1.1) and (3.6) motivate the following:

**Theorem 3.4.** *For any integers $p, m, n, k \geq 1$, if $n \geq mpk + 1$, then there exists a bijection between the set of $N_m^p$-separate $k$-subsets of $[\overline{n}]$ and the set of $N_1^p$-separate $k$-subsets of $[\overline{n}]$.*

We fail to produce such a bijection, and find that it remains a challenging open question. Now, we give several recurrence relations that the sequences $\mathcal{H}_{p,n}^{m,k}$ and $\mathcal{G}_{p,n}^{m,k}$ satisfy.

**Theorem 3.5.** *Let $p, m, k \geq 1$ be any integers. For $n \geq pm(k-1)$,*

$$\mathcal{H}_{p,n}^{m,k} = \mathcal{H}_{p,n-1}^{m,k} + \mathcal{H}_{p,n-p-1}^{m,k-1}, \tag{3.7}$$

*and for $n \geq m(pk+1)$,*

$$\mathcal{G}_{p,n}^{m,k} = \mathcal{G}_{p,n-1}^{m,k} + \mathcal{G}_{p,n-p}^{m,k-1}, \tag{3.8}$$

$$\mathcal{G}_{p,n}^{m,k} = \sum_{j\geq 0} (-1)^j \binom{m}{j} p^j (p+1)^{m-j} \mathcal{H}_{p,n-pm-j}^{m,k}, \tag{3.9}$$

*and for $n \geq mp(k-1)$,*

$$\mathcal{H}_{p,n}^{m,k} = \sum_{j\geq 0} (-1)^j \binom{m+j-1}{j} p^j \mathcal{G}_{p,n+pm-pj-j}^{m,k-j}. \tag{3.10}$$

**Proof.** To prove (3.7), let us consider $N_m^p$-separate $k$-subsets from $[n]$ which either contain the first object $x_1$ or do not. In the latter case, the number of such subsets is enumerated by $\mathcal{H}_{p,n-1}^{m,k}$. In the former case, the subsets do not contain the objects $x_{m+1}, x_{2m+1}, \ldots, x_{pm+1}$ of the set $A_1$ as defined in Section 2, note that the condition $n \geq mp(k-1)$ makes the restricted inequality condition of (2.1) in Proposition 2.1 redundant, so such subsets are counted by $\mathcal{H}_{p,n-p-1}^{m,k-1}$. Hence, (3.7) holds.

Using simple algebraic calculations we obtain that (3.8) holds.

Note that if $n \geq m(pk+1)$, there holds

$$\begin{aligned}
\mathcal{G}_{p,n}^{m,k} &= \operatorname*{Res}_x \frac{(1+x)^{n-pk-1}(1+(p+1)x)}{x^{k+1}} \\
&= \operatorname*{Res}_x \frac{(1+x)^{n+m-pk-1}}{(1+(p+1)x)^{m-1}} \left(p+1-\frac{p}{1+x}\right)^m x^{-k-1} \\
&= \sum_{j\geq 0}(-1)^j \binom{m}{j} p^j (p+1)^{m-j} \operatorname*{Res}_x \frac{(1+x)^{n+m-pk-j-1}}{(1+(p+1)x)^{m-1}} x^{-k-1} \\
&= \sum_{j\geq 0}(-1)^j \binom{m}{j} p^j (p+1)^{m-j} \mathcal{H}_{p,n-pm-j}^{m,k},
\end{aligned}$$

and if $n \geq mp(k-1)$, there holds

$$\begin{aligned}
\mathcal{H}_{p,n}^{m,k} &= \operatorname*{Res}_x \frac{(1+x)^{n+pm+m-pk-1}}{(1+(p+1)x)^{m-1}} x^{-k-1} \\
&= \operatorname*{Res}_x \frac{(1+x)^{n+pm-pk-1}(1+(p+1)x)}{x^{k+1}} \left(1+\frac{px}{1+x}\right)^{-m} \\
&= \sum_{j\geq 0}(-1)^j \binom{m+j-1}{j} p^j \operatorname*{Res}_x \frac{(1+x)^{n+pm-pj-j-p(k-j)-1}(1+(p+1)x)}{x^{k-j+1}} \\
&= \sum_{j\geq 0}(-1)^j \binom{m+j-1}{j} p^j \mathcal{G}_{p,n+pm-pj-j}^{m,k-j},
\end{aligned}$$

which prove (3.9) and (3.10). $\square$

The above theorem suggests that there should exist combinatorial proofs for (3.8)–(3.10). However, we fail to produce such proofs, and find them remaining challenging open questions.

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